

# Center of Mackey Functors

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## Abstract

This paper studies the center of the category of Mackey functor using the centers defined on monoidal categories. We define the category of Mackey functors in terms of span categories. Mackey functors became importance in the theory of representation of groups since groups have been studied using Mackey functors during the last 40 years. We see that the category of Mackey functors is a monoidal category and its monoids are Green functors. Center for Mackey functors is defined and using the cross  $G$ -sets, it was shown that centre of the category of Mackey functors is equivalent to the category of crossed  $G$ -sets. Some properties of the center of the category of Mackey functors are also studied in this paper.

**Keywords:** monoidal category, center, lax center, monoids, mackey functors.

## Introduction

The center and lax center of a monoidal category were defined and studied in unpublished work of Drinfeld and were studied in the papers (Joyal and Street, 1991) and (Day, Panchadcharam and Street, 2005). The center of a monoidal category was introduced in (Joyal and Street, 1991) in the process of proving that the free tortile monoidal category has another universal property. The center of a monoidal category is a braided monoidal category. The lax center of a monoidal category was considered under the name of “weak center” by P.Schauenburg (Schauenburg, 1994). Mackeyfunctors is an algebraic structure which behave like the induction, restriction and conjugation mappings in group representation theory. Mackey functors were first introduced by J.A. Green and A. Dress in the early 1970's as a tool for studying representations of finite groups and their subgroups. There are (at least) three equivalent definitions of Mackey functors for a finite group  $G$  (Bouc, 1997). The first one defines the Mackey functors as a poset of subgroups of a group  $G$  (Green, 1971). This is the most accessible definition of a Mackey functor for a finite group is expressed in terms of axiomatic relations which were developed by A. Green. The second one defines the Mackey functors in the sense of categorical way which was defined by A. dress (Dress, 1973). The third one defines the Mackey functors as modules over the Mackey algebra (Thevenaz and Webb, 1995). But they all amount to the same thing. In this paper we use the second definition which defines the Mackey functors in terms of categories. The purpose of this work is to study the center and the lax center of the category of Mackey functors. The main result of the paper is that the centre of the category of Mackey functors is equivalent to the category of crossed  $G$ -sets.

### 1. Definition of Mackey Functors

In this paper, we use the categorical version of the definition which was given by A. dress (Dress, 1973) to define Mackey functors. A Mackey functor over a ring  $R$  is a pair of functors  $M = (M_*, M^*)$  from the category of  $G$ -sets to the category of  $R$ -modules,  $R\text{-Mod}$ , so that  $M_*$  is covariant and  $M^*$  is contravariant,  $M_*(X) = M^*(X)$  for all finite  $G$ -sets  $X$  and such that the following axioms are satisfied:

1. For every pullback diagram of  $G$ -sets

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ \beta \downarrow & & \downarrow \gamma \\ Z & \xrightarrow{\delta} & U \end{array}$$

we have  $M^*(\delta)M_*(\gamma) = M_*(\beta)M^*(\alpha)$ .

2. For every pair of finite  $G$ -sets  $X$  and  $Y$ , we have  $M(X) \oplus M(Y) \rightarrow M(X + Y)$  is an isomorphism.

Now we will define the  $Spn(\mathcal{E})$  category (Benabou, 1967) to develop the properties of Mackey functors. Let  $\mathcal{E}$  be a finitely compact closed category. We define the compact closed category  $Spn(\mathcal{E})$  of spans in the category  $\mathcal{E}$ .

The objects of  $Spn(\mathcal{E})$  are the objects of the category  $\mathcal{E}$  and morphisms  $U \rightarrow V$  are the isomorphism classes of spans from  $U$  to  $V$  in the bicategory of spans in  $\mathcal{E}$ . A span from  $U \rightarrow V$  is a diagram of two morphisms with a common domain  $S$  as in the diagram:

$$(s_1, s_2): \begin{array}{ccc} & S & \\ \swarrow & & \searrow \\ U & & V \end{array}$$

The category  $Spn(\mathcal{E})$  becomes a monoidal category using the cartesian product in  $\mathcal{E}$  as follows:

$$Spn(\mathcal{E}) \times Spn(\mathcal{E}) \rightarrow Spn(\mathcal{E})$$

defined by  $(U, V) \mapsto U \times V$ .

## 2. Mackey Functors on Compact Closed Categories

Now we define the Mackey functor on a compact closed category using the span categories explained above. A Mackey functor  $M$  from a finitely compact closed category  $\mathcal{E}$  to the category  $Mod_k$  of  $k$ -modules consists of two functors

$$M_*: \mathcal{E} \rightarrow Mod_k, \quad M^*: \mathcal{E}^{op} \rightarrow Mod_k$$

satisfying the following conditions:

1.  $M_*(U) = M^*(U) = M(U)$  for all  $U$  in  $\mathcal{E}$ .
2. For all pullbacks in  $\mathcal{E}$ ,

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & V \\ \beta \downarrow & & \downarrow \gamma \\ U & \xrightarrow{\delta} & W \end{array}$$

the following square commutes and this square is called the *Mackey square*:

$$\begin{array}{ccc} M(P) & \xrightarrow{M_*(\alpha)} & M(V) \\ M^*(\beta) \uparrow & & \uparrow M^*(\gamma) \\ M(U) & \xrightarrow{M_*(\delta)} & M(W) \end{array}$$

3. For all coproduct diagrams

$$U \xrightarrow{i} U + V \xleftarrow{j} V$$

in  $\mathcal{E}$ , the diagrams

$$M(U) \xrightarrow{M_*(i)} M(U + V) \xleftarrow{M_*(j)} M(V)$$

and

$$M(U) \xleftarrow{M^*(i)} M(U + V) \xrightarrow{M^*(j)} M(V)$$

is a direct sum situation in the category  $Mod_k$ . By this conditions we get a simplified expression

$$M(U + V) \cong M(U) \oplus M(V).$$

A morphism  $\theta: M \rightarrow N$  of Mackey functors  $M$  and  $N$  is a family of morphisms  $\theta_U: M(U) \rightarrow N(U)$  for each  $U \in \mathcal{E}$  which gives two natural transformations

$$\theta_*: M_* \rightarrow N_* \text{ and } \theta^*: M^* \rightarrow N^*.$$

By the above definition, we showed that the Mackey functors become a category from a finitely compact closed category  $\mathcal{E}$  to the category of  $k$ -modules  $Mod_k$ . We denote this category by  $\mathbf{Mky}(\mathcal{E}, Mod_k)$ .

The following theorem gives a simplified version for  $\mathbf{Mky}(\mathcal{E}, Mod_k)$  of the category of Mackey functors from a finitely compact closed category  $\mathcal{E}$  to the category  $Mod_k$  of  $k$ -modules. We use an important theorem in the paper of H. Lindner (Lindner, 1976) to get this simplification.

**Theorem:**

The category  $\mathbf{Mky}(\mathcal{E}, Mod_k)$  of Mackey functors from a compact closed category  $\mathcal{E}$  to the category  $Mod_k$  of  $k$ -modules is equivalent to the category  $[Spn(\mathcal{E}), Mod_k]_+$  of co-product preserving functors from the category  $Spn(\mathcal{E})$  of the spans of  $\mathcal{E}$  to the category  $Mod_k$  of  $k$ -modules.

**Proof:**

Here we give a sketch proof for the above theorem. Let  $M$  be a Mackey functor from  $\mathcal{E}$  to  $Mod_k$ . Then we have a pair  $(M^*, M_*)$  such that  $M_*: \mathcal{E} \rightarrow Mod_k$ ,  $M^*: \mathcal{E}^{op} \rightarrow Mod_k$  and  $M(U) = M_*(U) = M^*(U)$ .

Now define a functor  $M: Spn(\mathcal{E}) \rightarrow Mod_k$  by  $M(U) = M_*(U) = M^*(U)$  and  $M(S_1, S_2)$  is given by  $M^*(S_1): M(U) \rightarrow M(S)$  and  $M_*(S_2): M(S) \rightarrow M(V)$ . we can show that  $M$  is well defined and becomes a functors. This functor will satisfies the conditions for a Mackey functor.

Conversely, let  $M: Spn(\mathcal{E}) \rightarrow Mod_k$  be a functor. Then we can define two functors  $M_*$  and  $M^*$  by referring the following diagrams:

$( )_*: \mathcal{E} \rightarrow Spn(\mathcal{E})$ ,  $( )^*: \mathcal{E}^{op} \rightarrow Spn(\mathcal{E})$ , and  $M: Spn(\mathcal{E}) \rightarrow Mod_k$  by putting  $M_*: M \circ (-)_*$  and  $M^*: M \circ (-)^*$ . Then we can show the necessary conditions easily. This completes the proof of the theorem.

### 3. Tensor Product and Closed Structure of $\mathbf{Mky}(\mathcal{E}, Mod_k)$ of the Category of Mackey Functors

Now we define a tensor product for the category  $\mathbf{Mky}(\mathcal{E}, Mod_k)$  of Mackey functors from the category  $\mathcal{E}$  to the category  $Mod_k$  of  $k$ -modules. This tensor product should be equivalent to the tensor product defined on the category  $[Spn(\mathcal{E}), Mod_k]_+$  of coproduct preserving functors from  $Spn(\mathcal{E})$  to  $Mod_k$  of  $k$ -modules by the above theorem.

The tensor product can be established as follows:

$$(M * N)(Z) = \int^Y M(Z \otimes Y^*) \otimes_k N(Y).$$

The Burnside functor  $J$  can be defined to be the Mackey functor on  $\mathcal{E}$  such that an object  $U$  of  $\mathcal{E}$  is a free  $k$ -modules on  $\mathcal{E}(1, U)$ . The Burnside functor becomes the unit for the tensor product of the category  $\mathbf{Mky}(\mathcal{E}, Mod_k)$  of Mackey functors from the category  $\mathcal{E}$  to the category  $Mod_k$  of  $k$ -modules. This convolution satisfies the necessary commutative and associative conditions for a symmetric monoidal category (Day, 1970).

The closed structure for the category  $\mathbf{Mky}(\mathcal{E}, Mod_k)$  of Mackey functors from the category  $\mathcal{E}$  to the category  $Mod_k$  of  $k$ -modules can be established using the Hom Mackey functor. The Hom Mackey functor is given by

$$\text{Hom}(M, N)(V) = \mathbf{Mky}(M(V^* \otimes -), N).$$

There is also another expression for this Hom Mackey functor, which is given by

$$\text{Hom}(M, N)(V) = \mathbf{Mky}(M, N(V \otimes -)).$$

Therefore, the category  $\mathbf{Mky}(\mathcal{E}, Mod_k)$  of Mackey functors from the category  $\mathcal{E}$  of a finitely compact closed category to the category  $Mod_k$  of  $k$ -modules becomes a symmetric monoidal closed category with the above established properties. Now we will show that the monoids of this symmetric monoidal closed category  $\mathbf{Mky}(\mathcal{E}, Mod_k)$  are the Green functors which are of one type of Mackey functors.

### 4. Green Functors

A Green functor  $A$  on  $\mathcal{E}$  is a Mackey functor, that is  $A: \mathcal{E} \rightarrow \text{Mod}_k$  is a co-product preserving functor, which equipped with a monoidal structure

$$\mu: A(U) \otimes_k A(V) \rightarrow A(U \otimes V)$$

and a morphism

$$\eta: k \rightarrow A(1).$$

Green functors become the monoids in the symmetric monoidal closed category  $\mathbf{Mky}(\mathcal{E}, \text{Mod}_k)$ .

## 5. Crossed G-Sets

We will consider the category  $\mathcal{E}/G_c$  of crossed  $G$ -sets. It was shown by Freyd-Yetter (Freyd-Yetter, 1989) that the category  $\mathcal{E}/G_c$  is a braided monoidal category. The objects are pairs  $(X, ||)$  where  $X$  is a  $G$ -set and  $||: X \rightarrow G_c$  is a  $G$ -set morphism. The tensor product of this category  $\mathcal{E}/G_c$  can be defined by

$$(X, ||) \otimes (Y, ||) = (X \otimes Y, || \otimes ||)$$

where  $|| (x, y) || = |x| |y|$ .

## 6. Center of a Mackey Functor

First we define the lax centre of a monoidal category  $\mathcal{E}$ . The lax center  $Z_l(\mathcal{E})$  of the category  $\mathcal{E}$  has objects of the form of ordered pairs  $(A, u)$ , where  $A$  is an object of the category  $\mathcal{E}$  and  $u$  is a natural family of morphisms  $u_B: A \otimes B \rightarrow B \otimes A$  such that the following two diagrams commutes:

$$\begin{array}{ccc} A \otimes B \otimes C & \xrightarrow{\quad} & A \otimes I \otimes A \\ & \searrow & \swarrow \\ & B \otimes A \otimes C & \end{array} \quad \begin{array}{ccc} & \xrightarrow{\quad} & \\ & \searrow & \swarrow \\ & A & \end{array}$$

Morphisms of the lax center  $Z_l(\mathcal{E})$  are morphisms in the monoidal category  $\mathcal{E}$  compatible with the morphism  $u$ .

The tensor product in  $Z_l(\mathcal{E})$  are defined by the following construction:

$$(A, u) \otimes (A, v) = (A \otimes B, w).$$

Using all the above constructions, we made the lax center  $Z_l(\mathcal{E})$  of a monoidal category  $\mathcal{E}$  is a monoidal  $V$ -category.

Now we define the centre  $Z(\mathcal{E})$  of a monoidal category  $\mathcal{E}$ . The objects of the center  $Z(\mathcal{E})$  are the objects  $(A, u)$  of the lax center  $Z_l(\mathcal{E})$ . The morphisms  $u$  of the center  $Z(\mathcal{E})$  are also same as the morphisms  $u$  of the lax center  $Z_l(\mathcal{E})$  but they are invertible. That is,  $u: A \otimes B \rightarrow B \otimes A$  and  $u: A \otimes B \leftarrow B \otimes A$ . So by this definition, we also made the center  $Z(\mathcal{E})$  of a monoidal category  $\mathcal{E}$  is a monoidal  $V$ -category in which each morphism is invertible.

The center  $Z(\mathbf{Mky}(\mathcal{E}, \text{Mod}_k))$  of the category of Mackey functors  $\mathbf{Mky}(\mathcal{E}, \text{Mod}_k)$  from a finitely compact closed category  $\mathcal{E}$  to the category  $\text{Mod}_k$  of  $k$ -modules consists of objects as pairs  $(M, \theta)$ , where  $M$  is a Mackey functor from the category  $\mathcal{E}$  to the category  $\text{Mod}_k$  of  $k$ -modules and  $\theta$  is a morphisms from the Mackey functors  $M$  to  $N$ , which is natural.

## Theorem

The centre  $Z(\mathbf{Mky}(\mathcal{E}, \text{Mod}_k))$  of the category of Mackey functors,  $\mathbf{Mky}(\mathcal{E}, \text{Mod}_k)$  from a finitely compact closed category  $\mathcal{E}$  to the category  $\text{Mod}_k$  of  $k$ -modules is equivalent to the category

$$\mathbf{Mky}(\mathcal{E}, \text{Mod}_k) / G_{\mathbf{Mky}(\mathcal{E}, \text{Mod}_k)}$$

of crossed  $G$ -sets.

## Proof:

Let we denote the category of Mackey functors as  $\mathcal{E}$ . Then,  $Z(\mathcal{E}) \rightarrow Z_l(\mathcal{E})$  is a faithful functor and then we get  $Z(\mathcal{E}) \rightarrow \mathcal{E}/G_c$ . Now we let  $||: A \rightarrow G_c$  be an object of the category of crossed  $G$ -sets  $\mathcal{E}/G_c$ . Then the corresponding object of the category  $Z_l(\mathcal{E})$  is  $(A, u)$  where

$$U_X: A \times X \rightarrow X \times A$$

we can easily show that  $u$  is invertible. This proves the theorem.

The above result is the main result of this paper. That is, the center of the category of Mackey functors from a finitely compact closed category is equivalent to the category of crossed  $G$ -sets of the category of Mackey functors. From this result, we can investigate the properties of the center of the category of Mackey functors through the properties of the category of crossed  $G$ -sets.

**Proposition**

Let  $(M, \theta)$  be an object of the lax center of the monoidal  $V$ -category  $\text{Mky}(\mathcal{E}, \text{Mod}_k)$  and let  $N$  be a Mackey functor from the category  $\mathcal{E}$  to the category  $\text{Mod}_k$  of  $k$ -modules (that is,  $N$  is an object of  $\text{Mky}(\mathcal{E}, \text{Mod}_k)$ ). Then,

$$\theta_{N^*}: M \otimes N^* \rightarrow N^* \otimes M$$

is an inverse for the morphism  $\theta_N: M \otimes N \rightarrow N \otimes M$ .

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